Diffusion from an entrance to an exit

by Michael E. Fisher

Asymptotic and exact solutions are derived from first principles by various methods for the moments of the number of steps or traversal time, etc., of a particle which diffuses, most specifically on a linear chain, to an exit site without previously leaving via an entrance site. The presentation is expository and uses standard methods.

1. Introduction

Rolf Landauer, in whose honor this note is penned, has posed the following problem: *A particle diffuses on a finite one-dimensional lattice or chain of sites, $x = 0, 1, \cdots, L - 1, L$, with unit lattice spacing, moving to the right or the left on each step with probability $\frac{1}{2}$. To start, the particle is inserted through the entrance site at $x_a = 0$ onto the initial site $x_i = 1$. It may eventually diffuse through the chain and, without first returning to the entrance, emerge at the exit site $x_e = L$. On average how many steps does the particle then take?* The proposed answer, for L large, is

$$n \approx \tfrac{1}{3}L^2. \tag{1}$$

This problem may be tackled and solved by fairly straightforward probabilistic or standard random-walk techniques, as set out, for example, in Feller's notable text [1]. The generality and power of these techniques are not, however, as widely appreciated as they might be. Furthermore, the explicit solution of a concrete problem is often informative, especially when, as here, the methods are of broad applicability. Accordingly, this paper offers two solutions which are developed "from scratch" and require little or no *a priori* knowledge.

The first solution starts by addressing the general problem of random walks on *any* lattice or graph which reach a given exit site, e, for the *first time* having avoided other sites $a, b, \cdots$; specialization to a uniform one-dimensional lattice and an asymptotic evaluation of a sum or integral yields (1) and further moments of the step distribution, such as

$$\langle \Delta n^2 \rangle \equiv \langle n^2 \rangle - \langle n \rangle^2 \approx \frac{2}{45} L^4/b^4, \tag{2}$$

where b^2 is the mean square size of a single step. In addition, explicit results are obtained for a general initial site x_i.

The second method tackles the finite one-dimensional problem head on and utilizes the eigenfunctions of a chain with a partial fraction decomposition and Fourier summations. It yields the *exact* answers for a walk which pauses or takes a "resting step" with probability $w_0 \equiv w$ but moves one site to the right or left with a probability $w_1 = \frac{1}{2}(1 - w)$ so that

$$b^2 = 1 - w \equiv \overline{w}. \tag{3}$$

The analysis gives

$$\langle n \rangle b^2 = \frac{1}{3}(L^2 - 1), \tag{4}$$

$$\langle \Delta n^2 \rangle b^4 = \frac{2}{45}\left[L^4 - 5\left(1 - \frac{3}{2}w\right)L^2 - \frac{15}{2}w + 4\right], \tag{5}$$

$$\langle \Delta n^3 \rangle b^6 = \frac{1}{945}[16L^6 + 42(2 + 3w)L^4 + 21(15w^2 - 15w - 4)L^2 - 16 + 189w - 315w^2], \tag{6}$$

and so on as desired! Note that these expressions all vanish identically on setting $L = 1$. This is correct, since the particle then exits as soon as it is inserted! Again, an arbitrary starting point, $x_i = \ell$, is readily handled.

Both methods adopt what might be called a *statistical mechanical formulation* by asking for the generating function or (grand canonical) partition function

$$G(z) := \sum_{n=0}^{\infty} g_n z^n, \qquad z \equiv e^{-\zeta}, \tag{7}$$

where g_n is the *weight* or *relative* probability of the desired walks, in this case walks executed by the particle in going from i to e, for the first time, in n steps without visiting $a, b, \cdots$. If, as here, the g_n can be read as probabilities of mutually exclusive events, then $G(1)$ represents the *total* probability, in this case, of exiting at e. Thus $g_n/G(1)$ is the probability distribution of the number of steps to exiting or, in other words, of traversal times, and so, as usual, one has

$$\langle n \rangle = z \frac{\partial}{\partial z} \ln G(z) \bigg|_{z=1} = -\frac{\partial}{\partial \zeta} \ln G \bigg|_{\zeta=0}, \tag{8}$$

$$\langle \Delta n^2 \rangle = \frac{\partial^2}{\partial \zeta^2} \ln G \bigg|_{\zeta=0}, \qquad \langle \Delta n^3 \rangle = -\frac{\partial^3}{\partial \zeta^3} \ln G \bigg|_{\zeta=0}, \tag{9}$$

with similar but more complex expressions for the higher moments. Note that we may suppose $|z| < 1$ or $\zeta > 0$ in all intermediate manipulations.

Apart from their common aim to calculate the generating function, the two methods presented below are independent, and a reader preferring to examine only the direct attack on the one-dimensional problem may skip directly to Section 4. Landauer and Büttiker [2] have addressed the problem using a device suggested by A. B. Pippard, but they obtain only the first moment(s) of exiting at e (and/or a). Gardiner has pointed out that the problem may be regarded as the limiting case of one analyzed in his book [3], but that connection is not immediately obvious and, naturally, his solution utilizes material developed in earlier parts of the book. H. Thomas and N. G. van Kampen have, in private correspondence with Landauer, reported their own solutions.

2. First visits with avoidance

Consider an arbitrary lattice or linear graph, finite or infinite, with sites or vertices, $i, j, k, \cdots$ on which a particle diffuses according to arbitrary transition probabilities but with no memory. (Of course, this is just a disguised Markov process.) Let $p_n(i \to j)$ be the probability *or*, more generally, statistical weight of all n-step walks from i to j. *What is the weight, $q_n(i \to j)$, of n-step walks from i to j which avoid, i.e., do not visit, any sites a, b, $\cdots$ belonging to a set A?* To answer this, and for its own interest, we also ask for $f_n(i \to e)$, *the weight of walks from i to e, avoiding sites in A, which reach e, the chosen exit, for the first time on the nth step.* In fact, one may regard the avoided set A as a set of irreversible exits (or traps) and merely enquire after those walks exiting at e.

To solve the problem we first relate the q_n and the f_n to the p_n; thence we will derive equations for the f_n in terms of the p_n, which we regard as known. Because there is no memory, the weight/probability of walks from i to j which go via site k is the product of the weights/probabilities of the walks from i to k and the walks from k to j. Consider, then, the unrestricted walks of total weight $p_n(i \to j)$ which go from i to j in n steps. If such walks avoid sites in A, their weight is just $q_n(i \to j)$. Otherwise, with weight $f_\ell(i \to a)$, the walk must reach some site, say a, in A for the first time on, say, step ℓ with $1 \le \ell \le n - 1$; thereafter the walk may proceed from a to j in $(n - \ell)$ steps which are quite unrestricted. Expressing this analysis in symbols and summing over all the disjoint possibilities yields the basic recursion relation

$$p_n(i \to j) = q_n(i \to j) + \sum_{\ell=1}^{n-1} \sum_{a \in A} f_\ell(i \to a) p_{n-\ell}(a \to j), \tag{10}$$

where i and j are not in A. Evidently, if the $f_\ell(i \to a)$ are known, this yields the desired q_n in terms of the p_n.

To obtain equations for the f_ℓ, apply the same argument but suppose that the final site $j \equiv e$ is an exit, i.e., belongs to A. Then $q_n(i \to j) \equiv f_n(i \to e)$ and (10) reduces to

$$p_n(i \to e) = f_n(i \to e) + \sum_{\ell=1}^{n-1} \sum_{a \in A} f_\ell(i \to a)\, p_{n-\ell}(a \to e), \tag{11}$$

where i is outside A. As e ranges over A, these relations provide a set of equations which, in fact, suffice to determine the f_n.

To take advantage of the convolution structure of (11), we introduce the generating functions (or *discrete Laplace transforms*)

$$P_{ij}(z) := \sum_{n=1}^{\infty} z^n p_n(1 \to j),$$

$$F_{ie}(z) := \sum_{n=1}^{\infty} z^n f_n(a \to e), \tag{12}$$

which, of course, contain all the known and desired information. If $|A|$ is the number of sites to be avoided, multiplying (11) by z^n and summing from $n = 1$ to ∞ yields the set of $|A|$ linear equations

$$F_{ie}(z) + \sum_{a \in A} P_{ae}(z) F_{ie}(z) = P_{ie}(z), \tag{13}$$

to be solved for the $|A|$ generating functions $F_{ie}(z)$. One may then return to (10), and for the generating function of the q_n one derives

$$Q_{ij}(z) = P_{ij}(z) - \sum_{a \in A} F_{ia}(z) P_{aj}(z). \tag{14}$$

The expressions (13) and (14) solve the problem posed, at least for finite $|A|$. When there are only a few sites to be avoided more explicit results are easy to obtain. Thus consider first

a. $|A| = 1 : A = \{a\}$.

The left side of (13) now entails only the number of n-step *returns* $r_n(i) := p_n(i \to i)$ which enter via

$$R_a(z) := \sum_{n=1}^{\infty} r_n(a) \equiv P_{aa}(z), \tag{15}$$

and lead simply to

$$F_{ia}(z) = P_{ia}(z)/[1 + R_a(z)], \tag{16}$$

and thence

$$Q_{ij}(z) = \frac{P_{ij}(1 + R_a) - P_{ia}P_{aj}}{1 + R_a}. \tag{17}$$

For the problem posed in the Introduction we need to avoid both the entrance site a and the exit site e, and so must consider

b. $|A| = 2 : A = \{a, e\}$.

Solution of the set (13) is trivial and leads to

$$F_{ie}(z) = \frac{P_{ie}(z)[1 + R_a(z)] - P_{ia}(z)P_{ae}(z)}{[1 + R_a(z)][1 + R_e(z)] - P_{ae}(z)P_{ea}(z)}, \tag{18}$$

and likewise for $F_{ia}(z)$; the formula for $Q_{ij}(z)$ is left to the reader.

The denominator in (18) represents the determinant $|\delta_{ab} + P_{ab}(z)|$. For $|A| > 2$, practical computations depend on the tractability of this determinant, which intimately reflects the structure of the avoided set A on the basic lattice or graph. Only if this has some relatively simple "shape" is one likely to obtain more implicit results. For $|A| = 2$ and a one-dimensional lattice or chain, however, it is not hard to go further, as we now show.

3. Evaluation of the avoidance formula

For symmetric, translationally invariant walks on an infinite d-dimensional lattice or corresponding torus, one has $p_n(i \to j) = p_n(j \to i)$ and $r_n(i) = r_n$ (independent of i); then (18) simplifies to

$$G_i(z) := F_{ie}(z) = \frac{P_{ie}(z)[1 + R(z)] - P_{ia}(z)P_{ae}(z)}{[1 + R(z)]^2 - P_{ae}^2(z)}. \tag{19}$$

To proceed we need expressions for the generating functions $P_{ij}(z)$ and $R(z) := P_{ii}(z)$.

Suppose we seek only results valid when the separation, L, between a and e is large; then, as will be seen, an asymptotic evaluation suffices. Two methods will be presented: the first supposes one knows the standard expression for the probability of diffusing from the origin, say at i, to the site j with coordinate $\mathbf{x}$ in n steps, namely, the Gaussian formula

$$p_n(\mathbf{x}) \approx e^{-x^2/2b^2 n}/(2\pi b^2 n)^{d/2}, \tag{20}$$

where, as in the Introduction, b^2 is the mean-square single-step length. For fixed $\mathbf{x}$ the summand in the generating function thus decays as $1/n^{d/2}$ and so, as $z \to 1$, which in view of (8) and (9) is what is required, the generating function $P_{\mathbf{x}}(z) := P_{ij}(z)$ diverges provided $d \le 2$. (The divergence reflects the fact that a walker is *certain* to exit eventually at e (a being avoided) when $d \le 2$; conversely, for $d > 2$ a walker may, with some probability, escape and never exit: See, e.g., Feller [1].) This divergence will cancel between numerator and denominator in (19) so that only the corresponding x-dependent amplitudes are actually needed. We may hence approximate the generating-function sum on n by an integral. For $d = 1$ we are thus led to

$$P_x(z) \approx \int_0^{\infty} dt\, e^{-\zeta t} e^{-x^2/2b^2 t}/(2\pi b^2 t)^{1/2}, \tag{21}$$

where, as before, $z = e^{-\zeta}$. The integral is a standard Laplace transform [4] whence

$$P_x(z) \approx e^{-\kappa(\zeta)|x|}/(2b^2\zeta)^{1/2}, \tag{22}$$

where we have introduced

$$\kappa(\zeta) = \sqrt{2}\, \zeta^{1/2}/b. \tag{23}$$

Finally, we may substitute in (19). Suppose, for added generality, that the particle starts at $i \equiv \ell$ lattice spacings from the entry site a and so at $L - \ell$ spacings from the exit e. Then one finds simply

$$G_\ell(z) = \frac{e^{-\kappa(L-\ell)} - e^{-\kappa\ell-\kappa L}}{1 - e^{-2\kappa L}} = \frac{\sinh \ell\kappa(\zeta)}{\sinh L\kappa(\zeta)}. \tag{24}$$

Expansion in powers of $\kappa \propto \zeta^{1/2}$ gives

$$\ln G_\ell(z) = \ln \frac{\ell}{L} - \frac{1}{3}\frac{L^2 - \ell^2}{b^2}\zeta + \frac{2}{45}\frac{L^4 - \ell^4}{b^4}\frac{\zeta^2}{2!} - \frac{16}{945}\frac{L^6 - \ell^6}{b^6}\frac{\zeta^3}{3!} + \cdots, \tag{25}$$

from which the results quoted in the Introduction, and more, follow via (8) and (9), with $\ell = 1 \ll L$. [The general coefficient of ζ^m in (25) is $(-)^m 2^{3m-1} B_m/m(2m)!$, where the B_m are Bernoulli's numbers.]

Now one might regard the approximation of the generating function sum by an integral as dubious, or not know (20) or be unable to perform the integral in (21). Then one may, alternatively, adopt a direct, knowledge-free route which starts with the basic single-step recursion relation for $p_{n+1}(i \to j) := p_{n+1}(\mathbf{x})$. To reach $\mathbf{x}$ in $n + 1$ steps, consider all n-step walks from i to the site at $\mathbf{x} - \mathbf{y}$ and add a single step of weight $p_1(\mathbf{y})$ to complete the walk. Summing over all possibilities gives

$$p_{n+1}(\mathbf{x}) = \sum_{\mathbf{y}} p_n(\mathbf{x} - \mathbf{y})p_1(\mathbf{y}). \tag{26}$$

Introducing the Fourier transforms

$$\hat{p}_n(\mathbf{k}) = \sum_{\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}}\, p_n(\mathbf{x}), \qquad \phi(\mathbf{k}) = \sum_{\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}}\, p_1(\mathbf{x}), \tag{27}$$

leads to the general solution

$$\hat{p}_n(\mathbf{k}) = [\phi(\mathbf{k})]^n, \tag{28}$$

when the initial site i is at the origin $\mathbf{x} = \mathbf{0}$. Multiplying by z^n and summing the resulting geometric progression gives the Fourier transform of the desired generating function, $P_{\mathbf{x}}(z)$, as

$$1 + \hat{P}_{\mathbf{k}}(z) = 1/[1 - z\phi(\mathbf{k})]. \tag{29}$$

Finally, Fourier inversion yields the exact expression

$$P_{\mathbf{x}}(z) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{1 - z\phi(\mathbf{k})}, \tag{30}$$

in which the integral runs over the Brillouin zone appropriate to the lattice studied.

We could, of course, derive (20) from this, but that step is not helpful here. Rather, note that for d not too large, the integral will be dominated by regions where the denominator is small. If one neglects (or averages suitably for) cases in which, for example, the walk can reach only one sublattice when n is odd and the other when n is even, then $\phi(\mathbf{k})$ has a unique maximum at $\mathbf{k} = 0$. Near this one has

$$1 - z\phi(\mathbf{k}) = \zeta + \frac{1}{2d} b^2k^2 + O(k^4, k^2b^2\zeta, \zeta^2) \tag{31}$$

when $z = e^{-\zeta} \to 1$, where, explicitly,

$$b^2 = \sum_{\mathbf{y}} y^2 p_1(\mathbf{y}). \tag{32}$$

These formulae are most easily checked for the one-dimensional, nearest-neighbor pausing walk defined in the Introduction, for which

$$\phi(k) = w + (1 - w)\cos k. \tag{33}$$

Substituting the approximation (31) in (30) and extending the region of integration gives

$$P_x(z) \approx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{\zeta + \frac{1}{2}b^2k^2} \qquad \text{for } d = 1. \tag{34}$$

This is valid asymptotically as $\zeta \to 0$. The integral is again standard [5] but is also easy to do by contour integration: It leads precisely back to (22) and (23), from which the answer (24) followed!

4. Diffusion on a chain

If one is allergic to the manipulation of generating functions or regards the general machinery deployed in Section 3 as too heavy for the problem in hand, a direct attack on the one-dimensional chain may appeal. Consider, first, the simplest case of a random walk which moves to the right or the left with probability $\frac{1}{2}$. If $q_n(x)$ is the weight of walks from i to the site j at x, the basic recursion relation is

$$q_{n+1}(x) = \frac{1}{2} q_n(x - 1) + \frac{1}{2} q_n(x + 1) \tag{35}$$

[which may be compared with (26)]. This has solutions of the form $q_n(x) \propto \lambda^n(\theta)e^{i\theta x}$ provided

$$\lambda(\theta) = \cos\theta. \tag{36}$$

Now, if the diffusing particle reaches the entrance a at $x = 0$ or the exit e at $x = L$, it is removed from the chain and should not be counted thereafter. This can be accomplished by imposing the boundary conditions

$$q_n(0) = q_n(L) = 0, \qquad \text{all } n. \tag{37}$$

These lead to the eigensolutions

$$q_n(x) \propto \lambda^n(k\pi/L)\sin(k\pi x/L), \tag{38}$$

with $k = 1, 2, \cdots, L - 1$, and hence the general solution is

$$q_n(x) = \sum_{k=1}^{L-1} c_k\lambda^n(k\pi/L)\sin(k\pi x/L). \tag{39}$$

To determine the amplitudes c_k we impose the general initial condition

$$q_0(x) = \delta_{x\ell} \qquad (\ell \neq 0, L), \tag{40}$$

where $\ell \equiv 1$ for the problem, as originally posed in the Introduction. On setting $n = 0$ in (39), multiplying by $\sin(j\pi x/L)$, summing on x and using (or easily proving) the orthogonality of the eigenfunctions, one obtains

$$c_k = (2/L)\sin(k\pi\ell/L). \tag{41}$$

Now we want g_n, which is the weight of walks exiting (for the first and only time!) at $x = L$ on their nth step. All such walks were at $x = L - 1$ on their $(n - 1)$th step and of these walks then exited. Hence we have

$$g_n = \frac{1}{2} q_{n-1}(L - 1) \qquad \text{for } n \geq 1, \tag{42}$$

and $g_0 = 0$ since $\ell \neq 0, L$ by supposition. The desired generating function for walks initially at $x = \ell$ is thus

$$G_\ell(z) := \sum_{n=0}^{\infty} z^n g_n = \frac{z}{L} \sum_{k=0}^{L-1} \frac{(-)^{k+1}\sin\theta\sin\ell\theta}{1 - z\cos\theta}, \qquad \theta = k\pi/L, \tag{43}$$

where we have used (39) and (41) with $\sin[k\pi(L - 1)/L] = (-)^{k+1}\sin(k\pi/L)$. Furthermore, the summation has been extended to include $k = 0$, which is valid since the summand vanishes at $k = 0$ provided $|z| < 1$, as is assumed henceforth.

The expression for $G_\ell(z)$ just obtained represents a formal answer to our problem. For $L = 1$ it yields $G_\ell(z) \equiv 0$, which is trivially correct; for $L = 2$, 3, and 4, one easily finds

$$G_1(z) = \frac{1}{2} z, \quad \frac{z^2}{4 - z^2}, \quad \text{and} \quad \frac{z^3}{8 - 4z^2}, \tag{44}$$

which results are not hard to check by direct consideration of walks on the corresponding short chains. For large L, however, it is imperative to perform the sum on k. Surprisingly, perhaps, it is possible to do this exactly!

The clue is to regard the denominator, $1 - z\cos\theta$, as proportional to a quadratic in $e^{i\theta}$ which can be factored, and then to use a partial fraction decomposition. The first step is most cleanly effected if one defines $\phi(z)$ through

$$e^{\pm\phi(z)} := \frac{1 \pm \sqrt{1-z^2}}{z} \equiv \frac{z}{1 \mp \sqrt{1-z^2}}, \tag{45}$$

or, what is equivalent,

$$\cosh\phi(z) := 1/z \equiv e^{\zeta}. \tag{46}$$

For $|z| < 1$ one can choose ϕ real and *positive*. Then the identities

$$\frac{iz\sin\theta}{1 - z\cos\theta} = \frac{1}{1 - e^{-\phi}e^{i\theta}} - \frac{1}{1 - e^{-\phi}e^{-i\theta}},$$
$$= \sum_{n=0}^{\infty} e^{-n\phi}(e^{in\theta} - e^{-in\theta}), \tag{47}$$

are easily checked. Now the summand in (43) is even in θ and so, if a factor $\frac{1}{2}$ is introduced, the limits on k may be further extended to $-L$ (to $L - 1$) and then, by periodicity, changed to $k = 0$ to $2L - 1$. Writing $-\sin \ell\theta = \frac{1}{2}i\,(e^{i\ell\theta} - e^{-i\ell\theta})$ and substituting in (43) then yields

$$G_\ell(z) = \frac{1}{2}\sum_{n=0}^{\infty} e^{-n\phi}(S^+_{n+\ell} - S^+_{n-\ell} - S^-_{n-\ell} + S^-_{n+\ell}), \tag{48}$$

where the $S^\pm_m$ are simple geometric sums which are easily evaluated. Explicitly one has

$$S^\pm_m = \frac{1}{2L}\sum_{k=0}^{2L-1} e^{ik\pi \pm imk\pi/L},$$
$$= \delta_{m,(2j+1)L} \quad \text{for} \quad j = 0, \pm 1, \pm 2, \cdots. \tag{49}$$

Using this in (48), taking care to avoid $n < 0$, and noting $0 < \ell < L$ leads finally to

$$G_\ell(z) = \sum_{j=0}^{\infty} e^{-(2j+1)L\phi}(e^{\ell\phi} - e^{-\ell\phi})$$
$$= \frac{e^{-L\phi}(e^{\ell\phi} - e^{-\ell\phi})}{1 - e^{-2L\phi}} = \frac{\sinh \ell\phi(z)}{\sinh L\phi(z)}, \tag{50}$$

a remarkably simple closed-form answer [which may be compared with the previous, asymptotic result (24)].

An expansion of $\ln G_\ell(z)$ identical to (25) holds, but with $b^2 = 1$ [as follows from (3) with $w = 0$] and ζ replaced by $\frac{1}{2}\phi^2$. Lastly, in order to expand in powers of ζ, as needed to evaluate the leading moments, one may use (45) or (46), which yield

$$\frac{1}{2}\phi^2 = \zeta + \frac{1}{3}\zeta^2 + \frac{2}{45}\zeta^3 + \cdots. \tag{51}$$

In our solution one has, of course, the freedom to vary the initial point at $x = \ell$. If one puts $\ell = L - 1$ and invokes right-left symmetry one sees that $G_{L-1}(z)$ represents the generating function for walks exiting at the entrance a. Thus one readily derives the corresponding exact mean "return time" and its dispersion, namely,

$$\langle n \rangle_a = \frac{2}{3}\left(L - \frac{1}{2}\right) \tag{52}$$

$$\langle \Delta n^2 \rangle_a = \frac{8}{45}\left(L^3 - \frac{3}{2}L^2 - \frac{3}{2}L + 1\right). \tag{53}$$

The result, $\langle n \rangle_a \approx \frac{2}{3}L$, has also been found by Landauer [2].

5. Pausing walk on a chain

It remains to treat the walk described in the Introduction, which pauses with probability $w \equiv 1 - \bar{w}$ on each step. The recursion relation (35) evidently becomes

$$q_{n+1}(x) = \frac{1}{2}\bar{w}q_n(x - 1) + wq_n(x) + \frac{1}{2}\bar{w}q_n(x + 1). \tag{54}$$

However, the results (39)–(41) remain valid if one replaces (36) with

$$\lambda(\theta) = w + \bar{w}\cos\theta. \tag{55}$$

[This may be compared to (33).] The expression (42) for g_n now requires a factor $\bar{w}$ on the right. In light of (55), the denominator in (43) becomes

$$1 - zw - z\bar{w}\cos\theta \equiv (1 - zw)(1 - \tilde{z}\cos\theta) \tag{56}$$

with

$$\tilde{z} := e^{-\tilde{\zeta}} = z\bar{w}/(1 - zw) \equiv \bar{w}e^{-\zeta}/(1 - we^{-\zeta}). \tag{57}$$

Consequently, the expression (43) for $G_\ell(z)$ remains correct if, in the second line, z is replaced with $\tilde{z}$. Likewise, (50) and (51) are still correct if z and ζ are replaced by $\tilde{z}$, and

$$\tilde{\zeta} = \frac{1}{\bar{w}}\zeta - \frac{w}{2\bar{w}^2}\zeta^2 + \frac{w(1 + w)}{6\bar{w}^3}\zeta^3 + \cdots. \tag{58}$$

Finally, utilizing (25) and replacing ζ with $\frac{1}{2}\phi^2(\tilde{z})$ yields

$$\ln G_\ell(z) = -\frac{1}{3}(L^2 - \ell^2)\frac{\zeta}{\bar{w}}$$
$$+ \frac{2}{45}\left[L^4 - \ell^4 - 5\left(1 - \frac{3}{2}w\right)(L^2 - \ell^2)\right]\frac{\zeta^2}{2!\bar{w}^2} + \cdots, \tag{59}$$

from which the detailed results quoted in (4)–(6) follow.

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References

1. W. Feller, *Probability Theory and Its Applications*, John Wiley & Sons, Inc., New York, 1950, Volume 1. See especially Chapters 12–14.
2. R. Landauer and M. Büttiker, "Diffusive Transversal Time; Effective Area in Magnetically Induced Interference," *Phys. Rev. B* **36** (1987); in press.
3. C. W. Gardiner, *Handbook of Stochastic Methods*, Springer-Verlag, Berlin, 1983; see Section 5.2.8, p. 142, and also in *J. Statist. Phys.* (1987); in press.
4. See, e.g., A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, McGraw-Hill Book Co., Inc., New York, 1954, Volume I, Equation 4.5 (27).

5. A. Erdélyi et al., *ibid.*, Equation 1.2 (11).

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