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Some Combinatorial Lemmas in Topology

Abstract: For many years it has been known that a combinatorial result, called the Sperner Lemma, provides an elegant proof of the Brouwer Fixed Point Theorem. Although the proof is elementary, its complete formal exposition depends upon the somewhat complicated operation of subdividing a simplex. Also, the proof does not show whether the Sperner Lemma can be derived from the Brouwer Fixed Point Theorem.

This central result of this paper is a combinatorial proposition, analogous to the Sperner Lemma, and applying to the *n*-cube, for which subdivision is a trivial operation. This Cubical Sperner Lemma follows immediately from the Brouwer Fixed Point Theorem and thus opens the possibility of other applications of topology to combinatorial problems. The question of such a topological proof is raised for another cubical analogue of the Sperner Lemma, due to Ky Fan, and for the Tucker Lemma, which is related to the antipodal point theorems. The Cubical Sperner Lemma of this paper implies the Tucker Lemma in 2-dimensions; this suggests that other connections joining these combinatorial results remain to be discovered.

1. Introduction

One of the most beautiful theorems of modern mathematics is the Brouwer Fixed Point Theorem, [1] both for its spare and weak hypotheses and for its wide and powerful applications. Its beauty is enhanced by an elegant proof due to Knaster, Kuratowski, and Mazurkiewicz, [2] which derives the result from a purely combinatorial lemma discovered by Sperner [3]. The following informal account of this argument for 2-dimensions will serve to motivate the investigations reported in this paper.

Consider a triangle T presented with barycentric coordinates $X = (x_0, x_1, x_2)$, where $x_0 \geqq 0$, $x_1 \geqq 0$, $x_2 \geqq 0$ and $x_0 + x_1 + x_2 = 1$. By a *proper labeling* of T, we shall mean the assignment of a label $L(X) = 0, 1,$ or 2 to each $X \varepsilon T$ such that if $L(X) = j$ then $x_j > 0$. (This condition says merely that the three vertices of T are labeled 0, 1, and 2, respectively, and that a point on an edge of T carries the label of one of the endpoints of the edge.) By a *subdivision* of T, we shall mean a decomposition of T into smaller triangles such that each edge of the decomposition is the edge of either one or two small triangles. The *mesh* of a subdivision is the largest diameter of any triangle of the subdivision.

Sperner Lemma. For any subdivision of T and any proper labeling, some small triangle of the subdivision carries a complete set of labels (0, 1, and 2) on its vertices.

This assertion is illustrated in Fig. 1, where the shaded triangle has a complete set of labels. The proof of the lemma follows easily from the fact that any small triangle which does not carry a complete set of labels has an even number of edges with 0 and 1 on their endpoints. Thus, if the lemma were false, the total number of such edges on the boundary of T would be even. However, this number is clearly odd.

For the derivation of the Brouwer Fixed Point Theorem from Sperner's Lemma, let $f : X \rightarrow Y = f(X)$ be a continuous function defined on T into T, that is, such that $y_0 \geqq 0$, $y_1 \geqq 0$, $y_2 \geqq 0$ and $y_0 + y_1 + y_2 = 1$ for all $Y = (y_0, y_1, y_2) = f(X)$.

Brouwer Fixed Point Theorem. There exists an $\bar{X} \varepsilon T$ such that $f(\bar{X}) = \bar{X}$.

Proof. The function f induces a proper labeling of T by means of the definition: Set $L(X) = j$ if $y_j \leqq x_j \neq 0$. If more than one label satisfies this condition, assign the smallest.

Choose a sequence of subdivisions of T such that

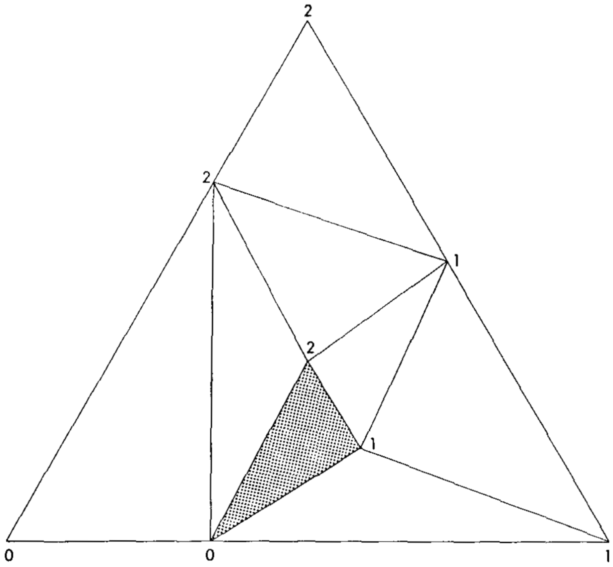


Figure 1 **The Sperner Lemma, in which shaded triangle carries a complete set of labels on its vertices.**

the associated sequence of meshes tends to zero. For each subdivision, choose a small triangle with a complete set of labels by Sperner's Lemma. For some subsequence, the barycenters of these small triangles converge to a point $\bar{X} \varepsilon T$. Since the meshes tend to zero, the three subsequences of the vertices of these triangles with labels 0, 1, and 2, respectively, also converge to $\bar{X}$. Hence, by the continuity of f and the definition of the labeling, $\bar{y}_j \leqq \bar{x}_j$ for $j = 0, 1, 2$. However, $\sum \bar{y}_j = \sum \bar{x}_j = 1$ and therefore $\bar{Y} = f(\bar{X}) = \bar{X}$. This completes the proof of the theorem.

Although this proof leaves little to be desired by way of clarity or simplicity, it raises two natural questions:

(1) The Brouwer Fixed Point Theorem follows directly from Sperner's Lemma. Is the reverse implication as direct?

(2) The formal description of the subdivision of a triangle (or, more generally, a simplex) is cumbersome. Does an analogue of the Sperner Lemma hold for the cube, for which subdivision is a trivial formal operation?

The central result of this paper is a combinatorial lemma which is a cubical analogue of Sperner's Lemma and which is equivalent to the Brouwer Fixed Point Theorem. This lemma will be stated and the equivalence proved in Section 2. A self-contained and complete proof of the lemma is given in Section 3. Certain related results and several open questions are discussed in Section 4. Some reflections on the significance of this type of result are presented in Section 5.

2. A Cubical Sperner Lemma

Let $I = (i_1, \cdots, i_n)$ denote an n-vector with all components integers. Relations between vectors are to hold in all components; for any integer i, let i stand for the vector $(i, \cdots, i)$ in vector relations. Let $I \leq I'$ denote $I \leqq I'$ and $I \neq I'$.

Cubical Sperner Lemma. Let Γ_p denote the set $\{I/0 \leqq I \leqq p\}$, for p a positive integer. Let L be a function defined on Γ_p into Γ_1 such that

$$0 \leqq I - 2L(I) + 1 \leqq p \quad \text{for all} \quad I \varepsilon \Gamma_p. \tag{1}$$

Then, in Γ_p there exist

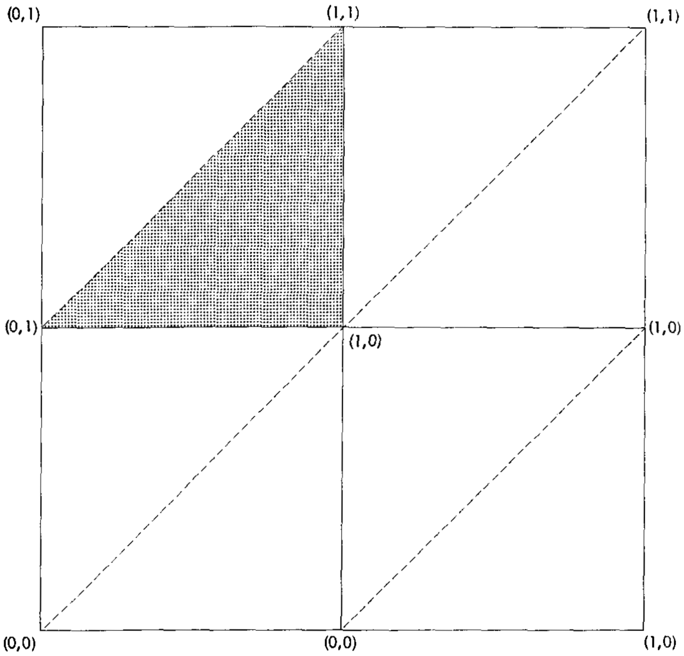
$$I_0 \leq I_1 \leq \cdots \leq I_m \leqq I_0 + 1 \tag{2}$$

such that

$$0 < L(I_0) + L(I_1) + \cdots + L(I_m) < m + 1. \tag{3}$$

As an informal explanation of the content of this lemma, Γ_p consists of the vertices of a subdivision of the n-cube into "small cubes." The function L assigns to each vertex I of Γ_p an n-component "label" $L(I) = (l_1, \cdots, l_n)$ with components 0 or 1. Condition (1) requires that $l_j = 0$ whenever $i_j = 0$ and $l_j = 1$ whenever $i_j = p$ and thus makes the labeling "proper." Condition (2) describes a set of I lying on a small cube in Γ_p; later it will be seen that this set spans an m-simplex in a canonical simplicial decomposition of the n-cube. Condition (3) asserts that, for no component j, are *all* of the labels on the vertices in this set equal to 0 or equal to 1. Figure 2 illustrates the lemma for $n = p = 2$.

Figure 2 **Cubical Sperner Lemma for the case $n = p = 2$.**



The labeling shown is proper. Triples satisfying (2) span the triangles indicated by dotted lines; the shaded triangle, among others, satisfies property (3).

For the derivation of the Brouwer Fixed Point Theorem from this lemma, let $X = (x_1, \cdots, x_n)$ denote a point in n-space and let $C = \{X \mid 0 \leqq X \leqq 1\}$ represent the unit n-cube. Furthermore, let $f : X \to Y = f(X)$ be a continuous function defined on C into C, that is, such that $0 \leqq Y \leqq 1$ for all $Y = (y_1, \cdots, y_n) = f(X)$. The function f induces a labeling of the points of C by means of the definition: Set $L(X) = (l_1(X), \cdots, l_n(X))$, where $l_j(X) = 0$ if $y_j \geqq x_j \neq 1$ and $l_j(X) = 1$ if $y_j \leqq x_j \neq 0$. If more than one label satisfies these conditions, assign the smaller in each component.

Now imagine C decomposed into p^n small cubes by means of the hyperplanes $x_j = i/p$ ($i = 1, \cdots, p-1$; $j = 1, \cdots, n$) perpendicular to the n coordinate directions. The vertices of this decomposition correspond in a natural way to the points of Γ_p. Precisely,

$$X = \left(\frac{i_1}{p}, \cdots, \frac{i_n}{p}\right) \leftrightarrow (i_1, \cdots, i_n) = I \in \Gamma_p.$$

By means of this correspondence, the labeling L of C defined by f induces a labeling of Γ_p. By the definition of L, this labeling clearly satisfies (1).

The hypotheses of the lemma being met, we may now choose, for each $p = 1, 2, \cdots$, a set

$$I_0^p \leq I_1^p \leq \cdots \leq I_m^p \leqq I_0^p + 1$$

(where m may vary with p) such that

$$0 < L(I_0^p) + L(I_1^p) + \cdots + L(I_m^p) < m + 1. \quad (4)$$

Since C is closed and bounded, we may also choose a subsequence $p_1, p_2, \cdots, p_l, \cdots$ such that the sequence $X_0^{p_l} = (1/p_l)I_0^{p_l}$ converges to a limit point $\bar{X} \in C$.

Now suppose $\bar{Y} = f(\bar{X}) \neq \bar{X}$, that is, $\bar{y}_j \neq \bar{x}_j$ for some j. If $\bar{y}_j < \bar{x}_j$, then by the continuity of f we have $y_j < x_j$ for all points X sufficiently close to $\bar{X}$. This means, for l sufficiently large,

$$l_j(I_0^{p_l}) + l_j(I_1^{p_l}) + \cdots + l_j(I_m^{p_l}) = m + 1.$$

Similarly, $\bar{y}_j > \bar{x}_j$ leads to the conclusion

$$l_j(I_0^{p_l}) + l_j(I_1^{p_l}) + \cdots + l_j(I_m^{p_l}) = 0.$$

Since both conclusions are ruled out by (4), $\bar{Y} = f(\bar{X}) = \bar{X}$ and the Brouwer Fixed Point Theorem is proved.

To derive the lemma from the Brouwer theorem, one needs the means to extend a mapping defined on the vertices of a subdivision of the cube to the entire cube. These are provided by the lemma [4]:

Lemma 1. Let $X \in C$. Then X has a unique representation

$$X = \lambda_0 I_0 + \cdots + \lambda_m I_m,$$

where $\lambda_0 > 0, \cdots, \lambda_m > 0$, $\lambda_0 + \cdots + \lambda_m = 1$ and

$$0 \leqq I_0 \leq \cdots \leq I_m \leqq 1.$$

Proof. First, we shall show that every $X \in C$ has such a representation by an explicit construction. Let $(j_1, \cdots, j_n)$ be a permutation of $(1, \cdots, n)$ such that $x_{j_1} \geqq \cdots \geqq x_{j_n}$. Define $I_0 = 0$ and $I_k = I_0 + E_{j_1} + \cdots + E_{j_k}$ for $k = 1, \cdots, n$, where E_j is the j^{th} unit vector. Clearly

$$0 = I_0 \leq I_1 \leq \cdots \leq I_n = 1.$$

Set $x_{j_0} = 1$ and $x_{j_{n+1}} = 0$, and define $\lambda_k = x_{j_k} - x_{j_{k+1}}$ for $k = 0, 1, \cdots, n$. Then all

$$\lambda_k \geqq 0, \qquad \lambda_0 + \cdots + \lambda_n = 1,$$

and

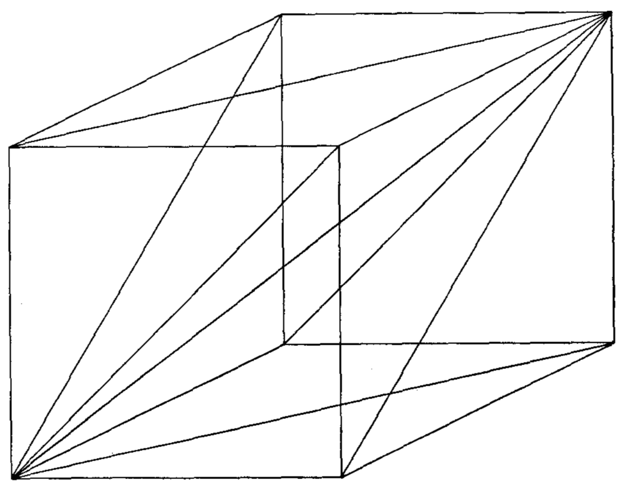
$$\begin{aligned}\lambda_0 I_0 + \lambda_1 I_1 + \cdots + \lambda_n I_n &= (x_{j_0} - x_{j_1})I_0 + \cdots + (x_{j_n} - x_{j_{n+1}})I_n \\ &= I_0 + x_{j_1}(I_1 - I_0) + \cdots + x_{j_n}(I_n - I_{n-1}) \\ &= x_{j_1}E_{j_1} + \cdots + x_{j_n}E_{j_n} = X.\end{aligned}$$

Dropping any I_k with zero λ_k, the required representation follows. It is clearly unique since $I_1, \cdots, I_n$ are linearly independent by their construction.

This result provides a canonical decomposition of the n-cube into (open) simplexes. The decomposition is illustrated for $n = 3$ in Fig. 3.

Corollary. Let $C_p = \{X \mid 0 \leqq X \leqq p\}$, for p a positive integer. Then every $X \in C_p$ has a unique representation,

Figure 3 **Canonical decomposition of the n-cube into open simplexes for the case $n = 3$.**



$X = \lambda_0 I_0 + \cdots + \lambda_m I_m,$ where

$\lambda_0 > 0, \cdots, \lambda_m > 0, \lambda_0 + \cdots + \lambda_m = 1$ and

$I_0 \leq \cdots \leq I_m \leqq I_0 + 1.$

Proof. Let $I = [X]$, that is, the vector composed of the integer parts of the components of X. Then $X - I \varepsilon C$ and Lemma 1 applies to provide the representation

$$X - I = \lambda_0 I_0' + \cdots + \lambda_m I_m'.$$

Setting $I_0 = I + I_0', \cdots, I_m = I + I_m'$, the Corollary follows.

• *Theorem 1.* The Brouwer Fixed Point Theorem for C_p implies the Cubical Sperner Lemma.

Proof. Let L be a labeling of Γ_p satisfying

$$0 \leqq I - 2L(I) + 1 \leqq p \quad \text{for all} \quad I \varepsilon \Gamma_p. \qquad (1)$$

The labeling is extended to a piecewise linear function on all C_p by the following definition: For $X \varepsilon C_p$, let $X = \lambda_0 I_0 + \cdots + \lambda_m I_m$ be the representation of the Corollary. Set

$$f(X) = X - 2[\lambda_0 L(I_0) + \cdots + \lambda_m L(I_m)] + 1$$

$$= \sum_{k=0}^{m} \lambda_k [I_k - 2L(I_k) + 1].$$

Then, clearly,

$$0 \leqq f(X) \leqq p$$

by (1), and hence f is a continuous function defined on C_p into C_p. Therefore, by the Brouwer Fixed Point Theorem, there exists an $\bar{X} \varepsilon C_p$ such that $f(\bar{X}) = \bar{X}$. For this $\bar{X}$,

$$\lambda_0 L(I_0) + \cdots + \lambda_m L(I_m) = \tfrac{1}{2}.$$

Hence

$$0 < L(I_0) + \cdots + L(I_m) < m + 1$$

and the lemma is proved.

3. A proof of the cubical lemma

As is often the case, an inductive proof of the lemma proceeds more smoothly when the assertion is strengthened. Let L be a (labeling) function defined Γ_p into Γ_1. For each $I \varepsilon \Gamma_p$, let $RL(I)$ denote the number of initial 0's in $L(I)$. Thus, RL is a (reduced labeling) function defined on Γ_p into the set $\{0, 1, \cdots, n\}$.

Strong Cubical Sperner Lemma. Let Γ_p denote the set $\{I/0 \leqq I \leqq p\}$, for p a positive integer. Let L be a function defined on Γ_p into Γ_1 such that

$$0 \leqq I - 2L(I) + 1 \leqq p \quad \text{for all} \quad I \varepsilon \Gamma_p. \qquad (1)$$

Then, in Γ_p there exist an odd number of sets

$$I_0 \leq I_1 \leq \cdots \leq I_n = I_0 + 1 \qquad (2')$$

such that

$$\{RL(I_0), RL(I_1), \cdots, RL(I_n)\} = \{0, 1, \cdots, n\}. \qquad (3')$$

Proof. A set in Γ_p satisfying (2′) will be called an *n-simplex*. A set in Γ_p satisfying

$$I_0 \leq I_1 \leq \cdots \leq I_{n-1} \leqq I_0 + 1 \qquad (2'')$$

will be called an $(n-1)$-*simplex*. Each n-simplex has $n + 1$ *faces* which are obtained by deleting any single vertex. Conversely, each $(n-1)$-simplex is the face of either one or two n-simplexes. This is seen by the following case analysis:

Case A. In (2″), all vertices $I_0, \cdots, I_{n-1}$ have their j^{th} component equal to 0. Then

$$I_0 \leq I_1 \leq \cdots \leq I_{n-1} \leq I_0 + 1 = I_0 + 1 \qquad (5)$$

is an n-simplex in Γ_p.

Case B. In (2″), all vertices $I_0, \cdots, I_{n-1}$ have their j^{th} component equal to p. Then

$$I_{n-1} - 1 \leq I_0 \leq \cdots \leq I_{n-1} = (I_{n-1} - 1) + 1 \quad (6)$$

is an n-simplex in Γ_p.

Case C. In (2″), all vertices $I_0, \cdots, I_{n-1}$ have their j^{th} component equal to q, where $0 < q < p$. Then both (5) and (6) are n-simplexes in Γ_p.

Case D. In (2″), no component of $I_0, \cdots, I_{n-1}$ is constant. Then, for some k, where $0 \leqq k < n - 1$,

$$I_{k+1} = I_k + E_r + E_s$$

for distinct unit vectors E_r and E_s. In this case,

$$I_0 \leq \cdots \leq I_k \leq I_k + E_r \leq I_{k+1} \leq \cdots \leq I_{n-1} = I_0 + 1$$

and

$$I_0 \leq \cdots \leq I_k \leq I_k + E_s \leq I_{k+1} \leq \cdots \leq I_{n-1} = I_0 + 1$$

are n-simplexes in Γ_p.

In Cases A and B, the $(n-1)$-simplex is called a *boundary* face; in Cases C and D, it is called an *interior* face. The four cases are illustrated in Fig. 4.

An n-simplex satisfying (3′) is said to be *complete*. An $(n-1)$-simplex satisfying

$$\{RL(I_0), RL(I_1), \cdots, RL(I_{n-1})\} = \{0, 1, \cdots, n-1\} \qquad (3'')$$

is said to be *complete*. Clearly every complete n-

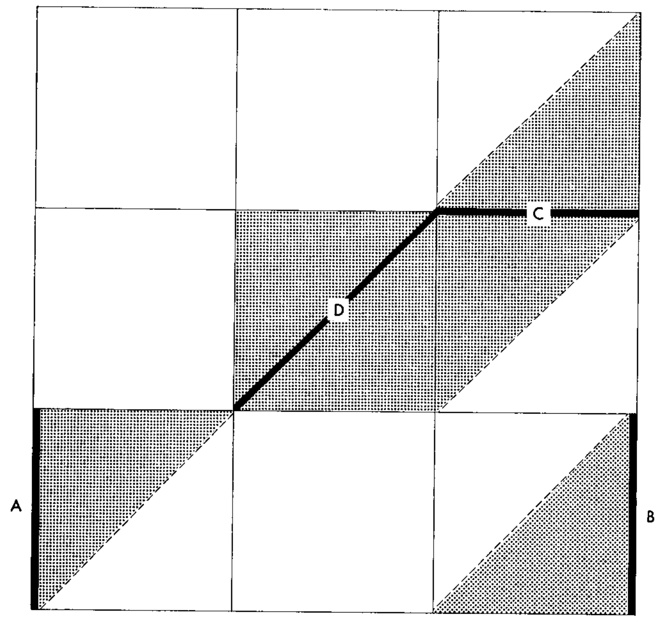


Figure 4 **Four cases for the (n — 1)-simplex in the Strong Cubical Sperner Lemma. Cases *A* and *B* are boundary faces; *C* and *D*, interior faces.**

simplex has exactly one complete face and every incomplete n-simplex has either zero or two complete faces. Hence *the parity of the complete n-simplexes equals the parity of the complete boundary faces* (since every complete interior face is counted twice and does not change the parity).

For a boundary face of type A, the reduced label $j - 1$ is missing, where j is a fixed index, $0 \leqq j \leqq n$. For a boundary face of type B, the reduced label j is missing, where j is a fixed index, $0 \leqq j \leqq n$. Therefore, the only complete boundary faces are of type B with $i_n = p$ for all vertices of the face. In Γ_p the vertices $I = (i_1, \cdots, i_n)$ for which $i_n = p$ satisfy the hypotheses of the lemma for dimension $n - 1$, if the n^{th} coordinates of both vertices and labels are deleted. (Note that $l_n = 1$ for all such vertices and hence no reduced label is n.) Since the lemma is clearly true for $n = 1$, an induction completes the proof.

4. Related lemmas and open questions

In this section, certain related combinatorial lemmas will be discussed and several open questions posed.

In Section 2, a direct argument showed that the Cubical Sperner Lemma follows from the Brouwer Theorem. The idea of this proof was extremely simple. By $f(I) = I - 2L(I) + 1$, each coordinate of I was increased or decreased by 1 if the corresponding component of $L(I)$ was 0 or 1, respectively. If a simplex of the subdivision has some component constant in all of its labels, then it is displaced one unit in that coordinate direction by f. Simplexes on the boundary of Γ_p are always moved into Γ_p since the labeling is proper. To deny the lemma would mean the existence of a continuous mapping of C_p into C_p with every point moved at least one unit.

Question 1. Is there an argument analogous to the preceding proof which will derive the original Sperner Lemma from the Brouwer Fixed Point Theorem?

The Fixed Point Theorem of Kakutani [5] is proved from the Brouwer Fixed Point Theorem using an approximation argument. This procedure seems unnatural and suggests:

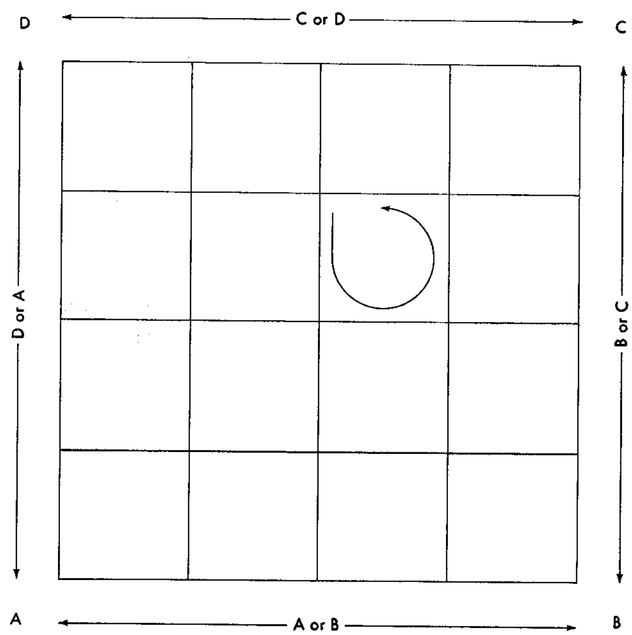
Question 2. Does there exist a combinatorial lemma which will serve as a basis for a direct proof of the Kakutani Fixed Point Theorem?

It is possible to prove the Cubical Sperner Lemma for $n = 2$ without resorting to a simplicial decomposition (triangulation) of the 2-cube (square). To show this argument, let $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, and $D = (0, 1)$ be the labels used; then the requirements for a labeling are shown in Fig. 5. The conclusion of the lemma may then be strengthened to:

There exists a small square of the subdivision with three distinct labels.

To prove this, orient the edges of each small square as shown in Fig. 5 and count each occurrence

Figure 5 **Requirements for labeling for proof of Cubical Sperner Lemma in case n = 2.**



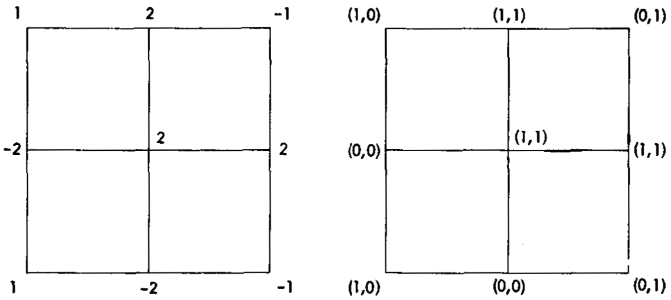


Figure 6 **Transformation of labeling to show connection between Strong Cubical Sperner Lemma and Tucker's Lemma, for $n = 2$.**

of AB (BA) as an edge of a small square as $+1$ (-1). If no small square has three distinct labels then this count is clearly zero. On the other hand, since interior edges are each counted twice with opposite orientations, the count may be made on the boundary alone. Here, the only contribution comes from the lower edge of Fig. 5 and the total count is clearly $+1$. This contradiction proves the lemma for $n = 2$.

Question 3. Is it possible to prove the Cubical Sperner Lemma without resorting to a simplicial decomposition of the n-cube?

Some years ago, A. W. Tucker discovered a combinatorial lemma which serves as the basis for a direct proof of the Borsuk-Ulam and Lusternik-Schnirelmann Antipodal Point Theorems. [6]

Tucker's Lemma. Let Γ_p denote the set $\{I/0 \leqq I \leqq p\}$, for p a positive integer. Let Δ denote the set $\{\pm 1, \cdots, \pm n\}$. There does not exist any function A defined on Γ_p into Δ such that

$$A(I) + A(I') \neq 0 \quad \text{if} \quad I \leqq I' \leqq I + 1 \tag{7}$$

and

$$A(I) + A(I') = 0 \quad \text{if} \quad I + I' = p \quad \text{and } not \quad 0 < I < p. \tag{8}$$

It is natural, in view of Section 2, to ask

Question 4. Is it possible to derive Tucker's Lemma from an Antipodal Point Theorem?

An interesting connection exists between the Strong Cubical Sperner Lemma and Tucker's Lemma, at least for $n = 2$. To show this, first establish the following $1-1$ correspondence between labels from Δ and labels from Γ_1:

$$+1 \leftrightarrow (1, 0) \qquad 2 \leftrightarrow (1, 1)$$

$$-1 \leftrightarrow (0, 1) \quad -2 \leftrightarrow (0, 0).$$

By means of this correspondence, any function A induces a labeling L, and vice versa. Notice that

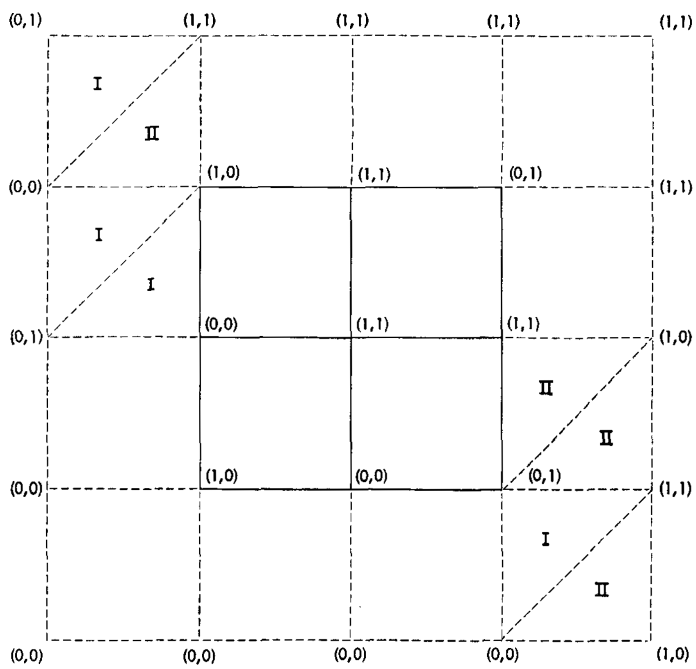
$$A(I) + A(I') = 0 \text{ if and only if } L(I) + L(I') = 1;$$

such labels $A(I)$ and $A(I')$ (or $L(I)$ and $L(I')$) will be called *complementary*. Condition (8) requires that antipodal labels on the boundary of Γ_p be complementary, while condition (7) requires that no small square carry complementary labels.

Assume a given labeling A of Γ_p that satisfies (8), and transform this into a labeling L. An example is given in Fig. 6. Clearly this labeling L need not be proper; however, by adding a border of squares to Γ_p it can be completed to a proper labeling in a number of ways. Carry out this completion so that antipodal vertices on the new boundary carry complementary labels and so that no new label is the complement of the adjacent label on the old boundary. Such a completion of Fig. 6 is shown in Fig. 7.

If any adjacent vertices in the old boundary are complements then (7) is not satisfied for A. Otherwise, the only occurrences of complementary labels in a small square of the border must be adjacent along a diagonal. Using only the diagonals of the canonical simplicial decomposition, these pairs have been indicated in Fig. 7. Each such diagonal is the face of two triangles; the third vertex in each such triangle is distinct from the labels of the diagonal. Hence, each such diagonal is the face of two triangles carrying one of the following four

Figure 7 **Completion of Figure 6 by adding border of squares to Γ_p and proper labeling.**



sets of labels (organized in two types):

Type I. $\{(1, 1), (0, 1), (0, 0)\}$ $\{(1, 0), (0, 1), (0, 0)\}$

Type II. $\{(0, 0), (1, 0), (1, 1)\}$ $\{(0, 1), (1, 0), (1, 1)\}$.

These occurrences have been labeled in Fig. 7 by their types. By the antipodal character of the labeling of the boundary, the total number of occurrences in the border is divisible by 4; by the complementary character of the two types, the total number of occurrences of Type 1 in the border is even.

On the other hand, Type 1 is exactly all complete sets of reduced labels that can be assigned to a triangle. The Strong Cubical Sperner Lemma asserts that the total number of occurrences of complete triangles in the square is odd. Hence there must be an odd number of occurrences of Type 1 in the original square. (These are left for the reader to find.) Hence condition (7) is not satisfied and Tucker's Lemma is proved. This leads us to ask:

Question 5. Is there a derivation of Tucker's Lemma from the Strong Cubical Sperner Lemma for all n?

Recently Ky Fan [7] has proved a different analogue of Sperner's Lemma for the n-cube.

Fan's Lemma. Let Γ_p denote the set $\{I/0 \leqq I \leqq p\}$ for p a positive integer. Let F denote a function defined on Γ_p into Γ_1 such that

$$0 \leqq I - 2F(I) + 1 \leqq p \quad \text{for all} \quad I \in \Gamma_p, \tag{1}$$

and

if I and I' are adjacent vertices of Γ_p, then $F(I)$ and $F(I')$ are either adjacent vertices of Γ_1 or coincide. (9)

Then there exists an odd number of small cubes of Γ_p which map *onto* Γ_1.

Question 6. Is there any relation between the Cubical Sperner Lemma of this paper and Fan's Lemma?

Question 7. Is Fan's Lemma equivalent to the Brouwer Fixed Point Theorem?

5. Conclusion

The various combinatorial results discussed in this paper all share their origins in topology. Although they have been stated in a geometric manner so as to facilitate their application, it is important to recognize that they are purely combinatorial in nature and can be stated without reference to any continuous geometric objects. Also, although they were devised for the purpose of proving certain theorems in topology, we have seen that the direction of this implication can be reversed in at least one important instance (Theorem 1). This opens the possibility of other applications of topology to combinatorial problems.

Another aspect of this paper which deserves comment is the use of cubical rather than simplicial complexes. As we have seen, cubical complexes have the crucial advantage of their ease of description in binary notation; for digital computation, therefore, they are a natural object to study. This advantage, however, is offset by the absence of an appropriate homology theory with a boundary operator suited to our purposes. One consequence of this fact for the present paper has been the use of simplicial decomposition to prove theorems which have purely cubical statements. Fan's Lemma, which has a cubical proof, offers some hope in this direction, but much remains to be done.

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References and footnotes

[1] This theorem, which states that every continuous mapping of a closed n-cell into itself has a fixed point, was first proved by L. E. J. Brouwer in 1912.

[2] B. Knaster, K. Kuratowski, and S. Mazurkiewicz, *Fundamenta Math.* **14**, 132 (1929).

[3] E. Sperner, *Abh. math. Sem. Hamburg*, **6**, 265 (1928).

[4] The simplical decomposition of the cube described in the lemma has been used previously by A. W. Tucker. It appears as Problem 3, p. 140, of *Introduction to Topology* by S. Lefschetz, Princeton University Press, 1949.

[5] S. Kakutani, *Duke Math. J.* **8**, 457 (1941).

[6] The lemma was stated and proved for $n = 2$ by A. W. Tucker, *Proceedings, Canadian Math. Congress*, p. 285 (Montreal 1945). His arguments for the general case, and the derivation of the antipodal point theorems, may be found on pp. 134–141 of *Introduction to Topology* by S. Lefschetz, Princeton University Press, 1949.

[7] Private communication.

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