C. W. Ho

Theory and Computer-aided Analysis of Lossless Transmission Lines

Abstract: A theoretical analysis of coupled and uncoupled lossless transmission lines is presented. A new method for deriving the conductance matrix **G** is described. Networks containing such lines have been simulated in the time domain and some results obtained from simulation of two example networks are given.

1. Introduction

This paper presents a theory of coupled and uncoupled lossless transmission lines and some practical numerical techniques for using a digital computer to simulate circuits containing such lines. The simulation problem has been a subject of much recent interest [1–4]; primary among the reasons for this interest are 1) the current trend toward large-scale integration (LSI), which has increased the density of circuits and brought them and their interconnecting transmission lines closer together than ever before; as a result, the electrical interactions among circuits and transmission lines are much more significant than they were with prior technologies; and 2) advances in sparse-matrix and implicit integration techniques have made it feasible for more circuits and elements to be analyzed simultaneously than was previously possible.

Although cut-off frequency and rise time specifications influence the importance of the dc resistance of transmission lines for each application, the resistance value is known to be negligible for many practical cases. In this paper we assume that the lines are lossless. The coupling material is allowed to be either homogenous or inhomogenous. [See Figs. 1(a) and (b).]

The paper is organized as follows: In Section 2 the properties of coupled lines are discussed. The conductance matrix **G** is derived in a new and compact form from transmission-line equations written in time-domain variables. The time-domain simulation problem is treated in Section 3, where a new and simple equivalent circuit is obtained. Some novel numerical techniques for integrating a set of differential-difference equations resulting from networks containing transmission lines are reported in Section 4 and numerical examples are given.

2. Properties and conductance matrix of coupled lines

To study the coupling effects of parallel transmission lines, we start with the self and mutual inductances and capacitances among the lines. Those parameters characterize the coupling in the vector form of the familiar telegrapher's equations as follows:

$$\frac{\partial \mathbf{e}(x,t)}{\partial x} + \mathbf{L}\,\frac{\partial \mathbf{i}(x,t)}{\partial x} = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{i}(x,t)}{\partial x} + \mathbf{C}\,\frac{\partial \mathbf{e}(x,t)}{\partial x} = 0\,, \tag{2}$$

where

$$\mathbf{e}(x,t) = [e_1(x,t), e_2(x,t), \cdots, e_n(x,t)]^{\mathrm{T}}$$

and

$$\mathbf{i}(x,t) = [i_1(x,t), i_2(x,t), \cdots, i_n(x,t)]^{\mathrm{T}}.$$

These vectors represent the line voltages and currents, respectively. (The superscript T is used here and throughout to indicate the transpose of a matrix or vector.) Distance and time are denoted by x and t, respectively, and **L** and **C** are symmetric inductance and capacitance matrices, respectively.

It can be easily shown that **L** is positive definite with all elements positive and that **C** is hyperdominant. (A hyperdominant matrix $\mathbf{C} = [c_{ij}]$ is defined as follows: $c_{ii} > 0$, $c_{ij} < 0$, and $c_{ii} > \sum_{j \neq i} |c_{ij}|$.)

Given a set of transmission lines as shown in Fig. 1 (b), the **L** and **C** matrices can be computed numerically with existing techniques [5] once the geometries of the lines and the dielectric constant and permittivity of the inhomogenous coupling material are specified.

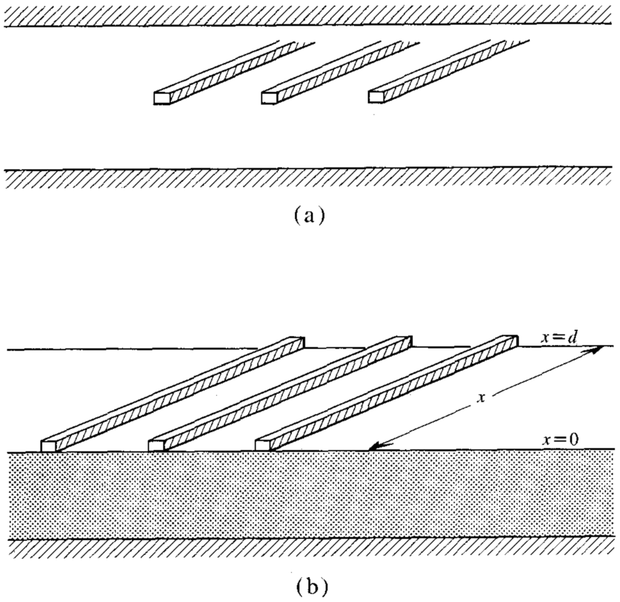

Figure 1 Coupled transmission lines. (a) In a homogenous medium, (b) in an inhomogenous medium.

It should be pointed out that the telegrapher's equations are inexact for accurately predicting transmission line voltages and currents if certain physical conditions exist. For example, if the wavelengths for the frequencies of interest are of comparable magnitude with the physical dimensions of the cross sections of the lines and the distances between them, then Maxwell's equations should be used directly. Nonetheless, for most practical applications, the telegrapher's equations yield good approximations.

Equations (1) and (2) can be manipulated to yield

$$\frac{\partial^2 \mathbf{i}(x,t)}{\partial x^2} = \mathbf{C}\,\mathbf{L}\,\frac{\partial^2 \mathbf{i}(x,t)}{\partial t^2}, \tag{3}$$

$$\frac{\partial^2 \mathbf{e}(x,t)}{\partial x^2} = \mathbf{L}\,\mathbf{C}\,\frac{\partial^2 \mathbf{e}(x,t)}{\partial t^2}. \tag{4}$$

If we now assume that conditions exist for a transverse electromagnetic (TEM) solution for the line voltage, we have the relation

$$\mathbf{e}(x,t) = \mathbf{f}(x - vt), \tag{5}$$

where v is the speed of wave propagation. Substituting (5) into (4), we obtain

$$(\mathbf{1} - \mathbf{L}\,\mathbf{C}\,v^2)\,\mathbf{f}''\,(x - vt) = \mathbf{0}.$$

It is thus clear that the speed of propagation is related to the eigenvalues λ_j of the matrix **L C** as follows:

$$v_j = \pm 1/\sqrt{\lambda_j}, \qquad j = 1,2,\cdots,n, \tag{6}$$

where n is the number of coupled lines and the the $\pm$ sign implies that the wave can travel in both the positive and negative directions along the x axis.

Now, taking any speed v_j, we use the method of characteristics [2] to combine

$$dx/dt = \pm\, v_j \tag{7}$$

with

$$d\mathbf{e}(x,t) = \frac{\partial \mathbf{e}(x,t)}{\partial x}\,dx + \frac{\partial \mathbf{e}(x,t)}{\partial t}\,dt, \tag{8}$$

$$d\mathbf{i}(x,t) = \frac{\partial \mathbf{i}(x,t)}{\partial x}\,dx + \frac{\partial \mathbf{i}(x,t)}{\partial t}\,dt \tag{9}$$

and Eqs. (1) and (2). This set of equations can be used to cancel some of the partial derivative terms to yield, for $+v_j$,

$$d[\mathbf{e}(x,t) + \mathbf{L}\,\mathbf{i}(x,t)\,v_j]/dt = \partial[(\mathbf{1} - \mathbf{L}\,\mathbf{C}\,v_j^2)\,\mathbf{e}(x,t)]/\partial t \tag{10}$$

and, for $dx/dt = -v_j$,

$$d[\mathbf{e}(x,t) - \mathbf{L}\,\mathbf{i}(x,t)\,v_j]/dt = \partial[(\mathbf{1} - \mathbf{L}\,\mathbf{C}\,v_j^2)\,\mathbf{e}(x,t)]/\partial t. \tag{11}$$

Consider now the coupled line system [Fig. 1(b)] under the following conditions. The n lines are initially at rest, and at $t = 0$, n voltage sources are applied to the input side $(x = 0)$. The magnitude of the first voltage source is equal to the first component of the jth eigenvector of the matrix **L C**, the second source is equal to the second component, and so on. Hence, line voltage $\mathbf{e}(x,t)$ is proportional to the jth eigenvector at all times and the right-hand sides of Eqs. (10) and (11) become zero.

Next, let us derive from Eq. (11) the time-dependent voltage-current relationships at the input and output ends of the lines. Integrating the left-hand side of (11) along the characteristic $dx/dt = -v_j$ from $x = d$ and $t = t - \tau_j$ to $x = 0$ and $t = t$, we obtain

$$\mathbf{e}(0,t) - \mathbf{L}\,\mathbf{i}(0,t)\,v_j = \mathbf{e}(d,t-\tau_j) - \mathbf{L}\,\mathbf{i}(d,t-\tau_j)\,v_j, \tag{12}$$

where τ_j is the delay for speed v_j $(\tau_j = d/v_j)$.

The right-hand side of Eq. (12) represents a wave that has travelled to the output side of the line and is returning to the input side. Since it takes $2\tau_j$ for a round trip, if $t < 2\,\tau_j$, the right-hand side of (12) is zero, and we have the relation

$$\mathbf{e}(0,t) = \mathbf{L}\,\mathbf{i}(0,t)\,v_j. \tag{13}$$

Equation (13) characterizes the voltage relationship at the input end of the transmission line set for the specified voltage sources and the time period $0 < t < 2\tau_j$. If we label this particular pair of voltage and current vectors $\mathbf{e}_j$ and $\mathbf{i}_j$, respectively, Eq. (13) can be written in the general form

$$[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n] = \mathbf{L}\,[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n][v_j], \tag{14}$$

where $[v_j]$ is a diagonal matrix with elements $v_1, v_2, \cdots, v_n$. Since the $\mathbf{e}_j(0,t)$ are defined to be the eigenvectors of matrix **L C**, we next want to show that the currents $\mathbf{i}_j$ given in Eq. (14) are actually the eigenvectors of matrix **C L**, or, equivalently

$$\mathbf{C\,L}\,[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n] = [\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n][\lambda_j]\,. \tag{15}$$

To show that Eq. (15) is valid, we manipulate Eq. (14) to yield

$$[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n] = \mathbf{L}^{-1}\,[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n][v_j]^{-1}$$

which implies that

$$\mathbf{C\,L}\,[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n] = \mathbf{C\,L\,L}^{-1}\,[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n][v_j]^{-1}$$
$$= \mathbf{L}^{-1}\,\mathbf{L\,C}\,[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n][v_j]^{-1}.$$

The fact that $[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n]$ are the eigenvectors of **L C** enables us to state, using Eq. (6), the relation

$$\mathbf{C\,L}\,[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n] = \mathbf{L}^{-1}\,[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n][v_j]^{-3}. \tag{16}$$

Equation (14) is then substituted into (16) and Eq. (15) is obtained. Hence we have established that if we drive the transmission line system with voltage sources having magnitudes equal to the components of an eigenvector of matrix **L C**, then, for $0 < t < 2\tau_j$, the magnitudes of the resultant line currents are equal to the component values of the eigenvector of matrix **C L** having the same eigenvalues as those of **L C**.

We now present the following theorem, which relates the eigenvectors of matrices **L C** and **C L**.

Theorem: Given positive definite and symmetric matrices **L** and **C**, we define the conductance matrix **G** as follows:

$$\mathbf{G} = \mathbf{L}^{-1}\,[\mathbf{L\,C}]^{\frac{1}{2}}, \tag{17}$$

where superscript ½ denotes the matrix obtained by taking the *positive* square root of its eigenvalues. Hence, we have $\mathbf{A} = \mathbf{A}^{\frac{1}{2}}\,\mathbf{A}^{\frac{1}{2}}$.

Further, let $\mathbf{E} = [\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n]$ and $\mathbf{I} = [\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n]$ where the $\mathbf{e}_j$ are the eigenvectors of **L C** and the $\mathbf{i}_j$ are the eigenvectors of **C L**. Then

(a) **G** is positive definite and symmetric, and

(b) $\mathbf{I} = \mathbf{G\,E}$. (18)

Proof: Consider the matrix $\mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}}$. The fact that it is symmetric can be shown by inspection. It is also positive definite because its quadratic form, for any vector **X**, is

$$Q = \langle \mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}}\,\mathbf{X},\mathbf{X}\rangle = \langle \mathbf{C}\,(\mathbf{L}^{\frac{1}{2}}\,\mathbf{X}),\,(\mathbf{L}^{\frac{1}{2}}\,\mathbf{X})\rangle \geq 0.$$

This is so because matrix **C** is positive definite. We can therefore decompose matrix $\mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}}$ into its canonical form as follows:

$$\mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}} = \mathbf{U}^{\mathrm{T}}\,\mathbf{D}\,\mathbf{U}, \tag{19}$$

where **U** forms an orthogonal transformation and **D** is diagonal with positive-valued elements. Using Eq. (19), we can manipulate matrix **L C** into the form

$$\mathbf{L\,C} = \mathbf{L}^{\frac{1}{2}}\,\mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}}\,\mathbf{L}^{-\frac{1}{2}} = \mathbf{L}^{\frac{1}{2}}\,\mathbf{U}^{\mathrm{T}}\,\mathbf{D}\,\mathbf{U}\,\mathbf{L}^{-\frac{1}{2}},$$

and, therefore,

$$(\mathbf{L\,C})^{\frac{1}{2}} = \mathbf{L}^{\frac{1}{2}}\,\mathbf{U}^{\mathrm{T}}\,\mathbf{D}^{\frac{1}{2}}\,\mathbf{U}\,\mathbf{L}^{-\frac{1}{2}}.$$

Using the definition of **G** given by Eq. (17), we have

$$\mathbf{G} = \mathbf{L}^{-1}\,(\mathbf{L\,C})^{\frac{1}{2}} = \mathbf{L}^{-\frac{1}{2}}\,\mathbf{U}^{\mathrm{T}}\,\mathbf{D}^{\frac{1}{2}}\,\mathbf{U}\,\mathbf{L}^{-\frac{1}{2}}. \tag{20}$$

The matrix **G** is seen to be symmetric by inspecting Eq. (20). The eigenvalues of **G** are the positive square roots of the eigenvalues of the matrix $\mathbf{L}^{\frac{1}{2}}\,\mathbf{C}\,\mathbf{L}^{\frac{1}{2}}$ which has just been shown to be positive. Therefore **G** is positive definite and symmetric.

To prove Eq. (18) we use the notation **I** and **E** to substitute for $[\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n]$ and $[\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n]$ in Eqs. (14) and (15), respectively. This yields

$$\mathbf{E} = \mathbf{L\,I}\,[v_j] \tag{21}$$

and

$$\mathbf{C\,L\,I} = \mathbf{I}\,[v_j]^{-2}. \tag{22}$$

Also, by definition, the matrix **E** contains the eigenvectors of matrix **L C**, and therefore,

$$\mathbf{L\,C\,E} = \mathbf{E}\,[v_j]^{-2}. \tag{23}$$

Multiplying the right-hand side of Eq. (22) by $[v_j]$ and using Eq. (21) to eliminate $\mathbf{L\,I}\,[v_j]$, we obtain

$$\mathbf{I} = \mathbf{C\,E}\,[v_j]. \tag{24}$$

We now prove Eq. (18) by first assuming its validity and then by showing that **G** is indeed defined by Eq. (17). We produce the matrix **G** as follows.

Put Eq. (18) into Eqs. (21) and (24) to obtain, respectively,

$$\mathbf{E} = \mathbf{L\,G\,E}\,[v_j] \tag{25}$$

and

$$\mathbf{G\,E} = \mathbf{C\,E}\,[v_j]. \tag{26}$$

From Eq. (26)

$$\mathbf{E}\,[v_j] = \mathbf{C}^{-1}\,\mathbf{G\,E}.$$

We substitute this into Eq. (25) to obtain

$$\mathbf{E} = \mathbf{L\,G\,C}^{-1}\,\mathbf{G\,E}, \tag{27}$$

which, when postmultiplied by $\mathbf{E}^{-1}$, yields

$$\mathbf{1} = \mathbf{L\,G\,C}^{-1}\,\mathbf{G} = \mathbf{L\,G}\,(\mathbf{L\,C})^{-1}\,\mathbf{L\,G}. \tag{28}$$

Then,

$$(\mathbf{L\,G})^{-1} = (\mathbf{L\,G})\,(\mathbf{L\,C})^{-1}$$

and after inverting both sides, we have

$$\mathbf{L}\,\mathbf{G} = \mathbf{L}\,\mathbf{C}\,(\mathbf{L}\,\mathbf{G})^{-1}$$

or

$$(\mathbf{L}\,\mathbf{G})^2 = \mathbf{L}\,\mathbf{C}.$$

It follows directly from $\mathbf{L}\,\mathbf{G} = (\mathbf{L}\,\mathbf{C})^{\frac{1}{2}}$ that

$$\mathbf{G} = \mathbf{L}^{-1}\,(\mathbf{L}\,\mathbf{C})^{\frac{1}{2}}, \tag{29}$$

which is Eq. (17), and thus the theorem is proved.

Consider now the general case in which the transmission-line system is driven by a set of arbitrary voltage sources. At any given time, the arbitrary voltage-source values can be considered as a linear combination of n sets of voltages, each set having its values proportional to the components of one eigenvector of $\mathbf{L}\,\mathbf{C}$; i.e., one column vector of $\mathbf{E}$. Hence, before the reflected wave returns to the input, the line voltages can be represented as

$$\mathbf{e}(x,t) = \mathbf{E}\,\boldsymbol{\alpha}(t). \tag{30}$$

This means that $\mathbf{e}(x,t)$ is a linear combination of the eigenvectors of $\mathbf{L}\,\mathbf{C}$ and that $\boldsymbol{\alpha}(t)$ is a time-varying weighting vector for any given distance x. It follows from the linearity property of Eqs. (1), (2) and the argument leading to Eq. (15) that the resultant line currents are determined by the same linear combinations, but with the eigenvectors of $\mathbf{C}\,\mathbf{L}$, or

$$\mathbf{i}(x,t) = \mathbf{I}\,\boldsymbol{\alpha}(t). \tag{31}$$

From Eqs. (30), (31) and (18), we conclude that

$$\mathbf{i}(x,t) = \mathbf{G}\,\mathbf{e}\,(x,t). \tag{32}$$

Therefore, under the stated conditions of analysis, the transmission line system behaves like a resistive n-port with its conductance matrix $\mathbf{G}$ defined by Eq. (17).

To conclude this section, the problem of synthesizing the conductance matrix $\mathbf{G}$ into a $(n+1)$-terminal resistive network is considered. It is well known that the necessary and sufficient condition for a given conductance matrix to be realizable as an $(n+1)$-terminal resistive network without transformers is that the matrix must be hyperdominant [6]. For the coupled lines, the conductance matrix is considered for the following three different configurations:

(*a*) *The coupling material is homogenous.* For this case the matrix $\mathbf{L}\,\mathbf{C}$ is diagonal. Hence $\mathbf{L}^{-1}$ is hyperdominant. From Eq. (17), $\mathbf{G}$ is the matrix product of a hyperdominant matrix and a diagonal matrix and is therefore hyperdominant. Thus $\mathbf{G}$ can be realized as a resistive network without transformers.

(*b*) *The coupling material has inhomogenous dielectric permittivity* ϵ *and magnetic permeability* μ. For this case a hyperdominant matrix $\mathbf{C}$ and a positive definite matrix $\mathbf{L}$ have been found such that $\mathbf{G}$ is not hyperdominant. An example is given by the matrices

$$\mathbf{L} = \begin{bmatrix} 4.0 & 3.8 & 3.5 \\ 3.8 & 5.0 & 4.8 \\ 3.5 & 4.8 & 5.0 \end{bmatrix} \mu\text{H/in.}$$

and

$$\mathbf{C} = \begin{bmatrix} 3.1 & -1.0 & -2.0 \\ -1.0 & 4.2 & -3.0 \\ -2.0 & -3.0 & 5.5 \end{bmatrix} \text{pF/in.},$$

where the corresponding $\mathbf{G}$ matrix is calculated to be

$$\mathbf{G} = \begin{bmatrix} 1.593 & -1.024 & -0.328 \\ -1.024 & 3.930 & -2.946 \\ -0.328 & -2.946 & 3.523 \end{bmatrix}$$

which is not hyperdominant because the summation of terms in the second row is less than zero.

Thus, in general, $\mathbf{G}$ cannot be realized as a resistive network without transformers.

(*c*) *The coupling material has homogenous permeability* μ *but inhomogenous dielectric permittivity* ϵ. This case is important in practice, for most coupled lines fall within this category. Homogenous μ implies that $\mathbf{L}^{-1}$ is hyperdominant—a fact that may be useful in proving the hyperdominance of $\mathbf{G}$. A proof or disproof of the hyperdominance of $\mathbf{G}$ under these conditions is not available to the author's knowledge and the problem seems to be open.

3. Simulation

For analysis in the time domain, an equivalent network consisting of lumped elements can be derived to replace the coupled lines. Let us observe for example a coupled line system at the input end $(x=0)$. Since the line voltage is the sum of the incoming voltage wave $\mathbf{e}^-$ and the outgoing voltage wave $\mathbf{e}^+$, while the line current is the difference between the two current waves, $\mathbf{i}^+$ and $\mathbf{i}^-$,

$$\mathbf{e}(0,t) = \mathbf{e}^+(0,t) + \mathbf{e}^-(0,t)\,, \tag{33}$$

$$\mathbf{i}(0,t) = \mathbf{i}^+(0,t) - \mathbf{i}^-(0,t)\,. \tag{34}$$

If we define $\mathbf{R}$ to be the inverse of $\mathbf{G}$, then from (32)

$$\mathbf{e}^+(0,t) = \mathbf{R}\,\mathbf{i}^+(0,t), \tag{35}$$

$$\mathbf{e}^-(0,t) = \mathbf{R}\,\mathbf{i}^-(0,t). \tag{36}$$

Multiply Eq. (34) by $\mathbf{R}$ and use Eqs. (35) and (36) to obtain

$$\mathbf{R}\,\mathbf{i}(0,t) = \mathbf{e}^+(0,t) - \mathbf{e}^-(0,t).$$

Substitution of this equation into (33) yields

$$\mathbf{e}(0,t) = \mathbf{R}\,\mathbf{i}(0,t) + 2\mathbf{e}^-(0,t)\,. \tag{37}$$

The equivalent circuit will be based on Eq. (37). Note in Eq. (37) that the term $2\mathbf{e}^-(0,t)$ denotes the voltage wave that starts from the receiving end, $x = d$, and reaches the sending end. The voltage wave that enters the receiving end can be derived from Eqs. (33) - (36) to obtain

$$2\mathbf{e}^+(d,t) = \mathbf{e}(d,t) - \mathbf{R}\,\mathbf{i}(d,t). \tag{38}$$

Since there are different delays due to different speeds of propagation, we need to decompose the voltage given by Eq. (38) into its components. This is achieved by using Eq. (30) to multiply the right-hand side of Eq. (38) by $\mathbf{E}^{-1}$. Each element of the resultant vector, which collectively represent the magnitudes for each different speed, is then delayed by different amounts. On reaching the sending end, the resultant voltages are then obtained by combining all the components by using $\mathbf{E}$. We therefore have

$$2\mathbf{e}^-(0,t) = \mathbf{E}\,\{\mathbf{E}^{-1}[\mathbf{e}(d,t) - \mathbf{R}\,\mathbf{i}(d,t)]\}\ (t-\tau)\,, \tag{39}$$

where $(t-\tau)$ is used as an operator that changes the argument of the ith element of the vector in brackets from t to $t-\tau_i$.

Upon substituting Eq. (39) into (37), there results

$$\mathbf{e}(0,t) = \mathbf{R}\,\mathbf{i}(0,t) + \mathbf{E}\{\mathbf{E}^{-1}[\mathbf{e}(d,t) - \mathbf{R}\,\mathbf{i}(d,t)]\}\ (t-\tau)\,. \tag{40}$$

Similarly, we have

$$\mathbf{e}(d,t) = -\mathbf{R}\,\mathbf{i}(d,t) + \mathbf{E}\{\mathbf{E}^{-1}[\mathbf{e}(0,t) + \mathbf{R}\,\mathbf{i}(0,t)]\}\ (t-\tau)\,. \tag{41}$$

Equations (40) and (41) can be considered as the branch constitutive relations for the voltage sources shown in Fig. 2. These two equations can be programmed straightforwardly to simulate coupled line systems in the time domain.

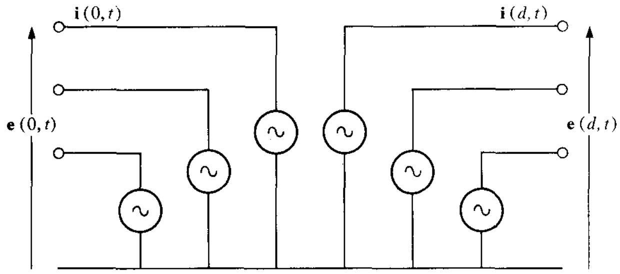

Figure 2 An equivalent network.

4. Programming considerations and numerical examples

To simulate circuits containing transmission lines in the time domain, it is necessary to account for the fact, shown in the last section, that delays are introduced into the circuit equations. The circuit equation thus becomes a set of differential-difference equations. In this section, our attention is focused on the numerical techniques used to control the step sizes for integrating such equations to achieve accuracy and stability.

In using multiple-step implicit integration formulae, the time delays in the equations in effect introduce additional multiple back steps. The resultant difference equation is, in general, not absolutely stable. An analytical approach that assures numerical stability appears to be difficult to achieve. However, practical experience has shown that the following techniques are effective in controlling time steps.

(*a*) *The look back feature:* Before accepting a time step h in the variable step size integration scheme, for each delay τ_i of every transmission line system, we go back τ_i seconds to get the previous time step h_i. The size of h is then reduced to $k\,h_i$ if $h > k\,h_i$, where k is a positive number with typical values ranging from 2 to 4.

(*b*) *The look ahead feature:* In variable step size integration schemes, the step sizes are usually determined by controlling the truncation errors to be within a given bound. The values of transmission line voltage sources are known in advance because they are the values from the reflected voltage waves. Hence, to determine whether a time step is too large for the transmission line elements, the equivalent voltage-source values at the new time point are first estimated by extrapolation using the back time points. The estimated values are then compared to the known values previously stored to determine if the truncation error is allowable. The step size can therefore be reduced if necessary.

(*c*) *Overshoot considerations:* An overshoot occurs if and when the magnitude of the time step exceeds some of the delays. It is easily seen that if overshoot occurs for some delays in the network, the techniques discussed in part (a) and part (b) become ineffective for those delays. Nevertheless, experience has shown that overshoot can be permitted in most practical cases for the sake of saving computation time if the starting step size is small, if the techniques in part (a) and (b) are used whenever possible, and if the following iteration scheme is implemented. For those delays smaller than the time step, the corresponding values for transmission line voltage sources are first estimated by extrapolation. The result is used to solve for the circuit variables. Then after the first and each subsequent Newton iteration those values can be and therefore are updated by interpolation. The process continues until convergence is observed for the time step.

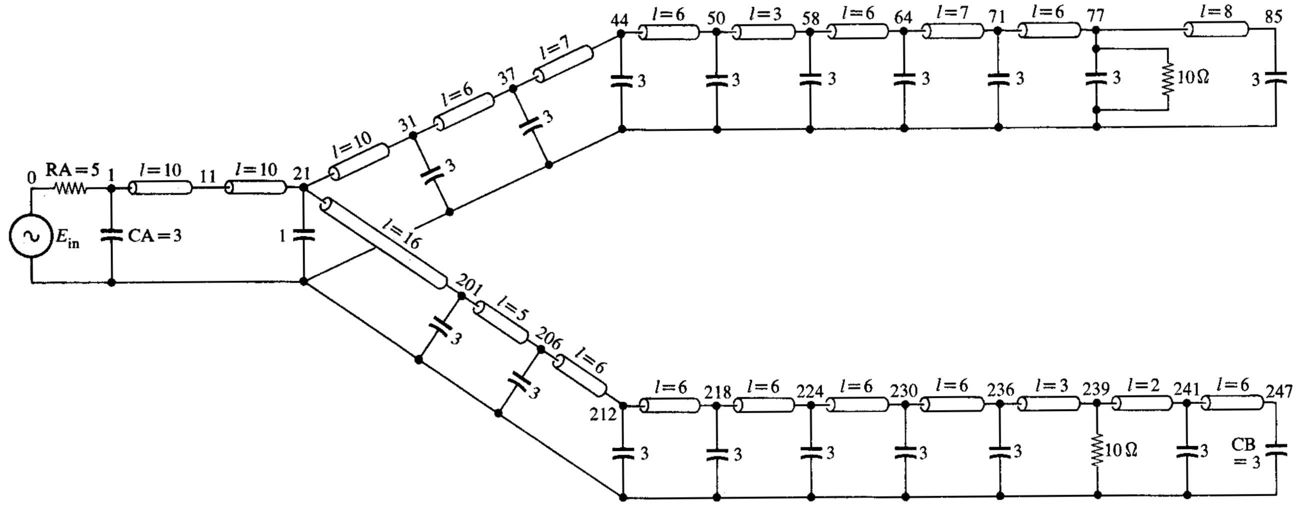

Figure 3 A network containing cascaded transmission lines.

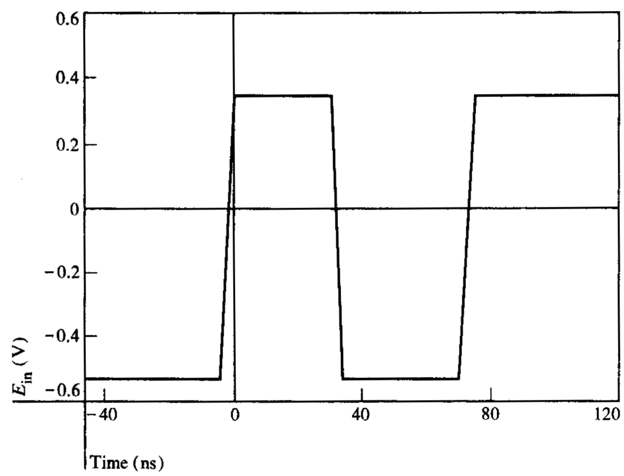

Figure 4 Voltage source $E_{in}(t)$.

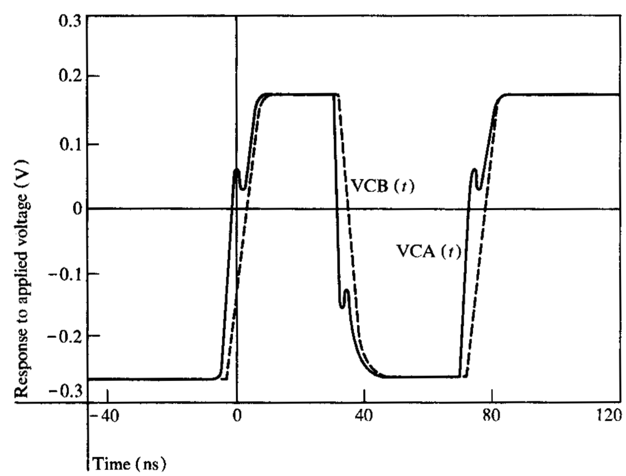

Figure 5 Voltages VCA(t) and VCB(t).

The following examples illustrate some of the techniques discussed in the paper.

Example 1. The network shown in Fig. 3 is driven by a voltage source $E_{in}(t)$ shown in Fig. 4. The transmission lines are all lossless. The inductance and capacitance of the lines are $L = 0.00138$ μH/in. and $C = 0.198$ pF/in., respectively. The length in inches of each line is also shown in Fig. 3. The network is to be analyzed from $t = -50$ ns to $t = 125$ ns and equilibrium is assumed to exist at the start of the analysis. From the inductance and the capacitance per unit length, the delays of the lines are calculated to be between 0.03306 ns and 0.2645 ns, respectively. The network is first analyzed by using the techniques discussed in this section with no limitation placed on the maximum step size. The result is that 304 time steps are taken with 632 Newton passes. The minimum and maximum time steps are 0.0066 ns and 8.5 ns, respectively. The network is analyzed again by limiting the time step size to be equal to the minimum delay of 0.03306 ns. The result is that 5182 time steps are taken with an equal number of Newton passes. The plots of the voltages across the capacitors CA and CB in the network are shown in Fig. 5. It is therefore seen that substantial saving of computation time is achieved by the techniques presented if the step size is not limited to be smaller than the delays.

Example 2. A network containing a system of three coupled transmission lines in a homogenous medium is shown in Fig. 6 where the inductance and the capacitance matrices for the coupled lines are:

$$\mathbf{L} = \begin{bmatrix} 0.00793 & 0.00301 & 0.00105 \\ 0.00301 & 0.00984 & 0.00301 \\ 0.00105 & 0.00301 & 0.00793 \end{bmatrix} \mu\text{H/in.},$$

and

$$\mathbf{C} = \begin{bmatrix} 4.72 & -1.42 & -0.085 \\ -1.42 & 4.23 & -1.42 \\ -0.085 & -1.42 & 4.72 \end{bmatrix} \text{pF/in.}$$

The length of the line is 11 inches. The delay for the transmission line is calculated to be 2 ns. The input voltage $E_{\text{in}}(t)$ is shown in Fig. 7. The network is analyzed from $t = 0$ to 40 ns with the use of a variable-step, variable-order integration scheme. A total of 488 time steps are taken. The plots of $\text{VR1}(t)$ and $\text{VR2}(t)$ are also shown in Fig. 7.

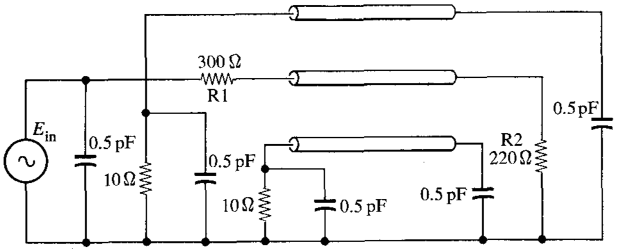

Figure 6 A network containing coupled transmission lines.

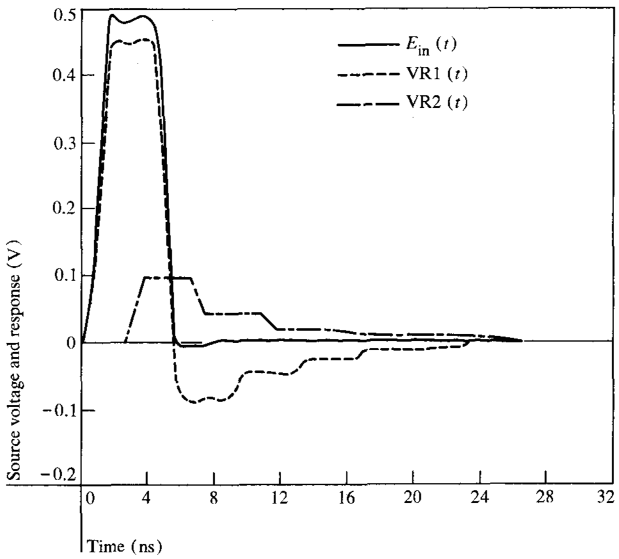

Figure 7 Voltages $E_{\text{in}}(t)$, $\text{VRI}(t)$ and $\text{VR2}(t)$.

5. Conclusions

The conductance matrix **G** for coupled transmission lines has been derived in a new and compact form and some of its properties have been discussed. The matrix **G** can be readily computed by using its relationship to both the capacitance and the inductance matrices.

The network simulation problem has been considered. A simple equivalent circuit for the transmission lines in the time domain has been obtained such that writing a simulation program is a straightforward matter. A substantial amount of computer time can be saved by using the numerical techniques shown in Section 4 to analyze circuits containing transmission lines in the time domain. In particular, the number of computing passes was reduced by a factor of 8 for the first example given.

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The author is located at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598.